Linear algebra notes

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# Determinant

A determinant is a number:

Note a determinant always has the same number of rows and columns, in other words, it is a square.

## Cofactor expansion

The remaining part after deleting the i-th row and j-th column is called a determinant ’s minor, denoted by :

The associated cofactor is then defined as . It is found that can be expanded according to a certain column or row. For instance, the expansion according to the 1st column is the following:

## Triangle determinant (determinant of a triangle matrix)

If is the determinant of a triangle matrix, then:

## Properties

* A determinant multiplied by a number is equivalent to the scenario in which the elements in a certain row or column are all multiplied by , i.e.,

Note this is very different from matrix-number multiplication.

* Interchanging two columns or rows changes the determinant’s sign
* If all elements in one row (column) are times that of the other, the determinant is 0
* If all elements in one row (column) can be broken to the sum to two numbers, i.e., , then:
* Multiplying one row (column) by and adding it to another row (column) does not change the determinant’s value

## [Cramer’s rule](https://en.wikipedia.org/wiki/Cramer%27s_rule)

It’s better to explain Cramer’s rule via one example – assume the following system of linear equations:

Then the solution is:

Cramer’s rule is quite computationally expensive and hence is not adopted as the major algorithm for solving the systems of linear equations.

## [Relation to the area of a parallelogram or the volume of a parallelepiped](https://mathinsight.org/relationship_determinants_area_volume)

### Relation to the area of a parallelogram

The area of a parallelogram spanned by tow vectors and is the magnitude of :

Assume bothand lie in the same plane so that , hence:

Therefore, the area of the parallelogram is given:

### Relation to volume of a parallelepiped

The volume of a parallelepiped spanned by three vectors , , and is the magnitude of :

Hence, the volume of a parallelepiped can be expressed as:

# Matrix

Unlike the determinant which is essentially a number, the matrix is a table of numbers. The identity matrix (equivalent to 1 in numbers) is:

## Addition and number multiplication

### Properties of matrix addition and number multiplication

Note multiplying a matrix by a number is equivalent to multiplying each of its elements by this number. This is different from determinant-number multiplication.

## Matrix multiplication

Let , then each element in is the inner product of the row vectors in and column vectors in :

Here, and are the row and column vectors of and , respectively.

### Properties of matrix multiplication

Please note, even though and **,** it’s still possible that .

### Time complexity

Matrix multiplication is very computationally expensive. A matrix multiplied by another matrix has a time complexity of . One trick is that if the matrix multiplication is followed by vector multiplication, perform the vector multiplication first as it reduces the time complexity substantially.

## Matrix transpose

### Properties of matrix transpose

Given that , the last property is exceptionally interesting. It can be generalized to finite number of matrices:

Both and are symmetric matrices.

Any square matrix can be decomposed to the sum of a symmetric and an anti-symmetric matrix:

## Matrix inverse

Definition: if , then is invertible and the inverse of is.

to be invertible iff . Under such a circumstance:

Here is’sadjugate matrix. If , then has the following form:

In other words, is the transpose of the matrix constructed by the cofactors of each element of .

### Properties of matrix inverse

The last property is very similar to matrix transpose. Its quick proof is in the following:

Similar to matrix transpose, this property can be generalized to finite number of matrices:

## Block matrix

Let and , then the following holds:

Here . In other words, the rule is the same as ordinary matrix multiplication.

### Some special cases

If , , then the following rules hold:

Similar to multiplication, these rules are the same as the corresponding ordinary matrix operations.

## Elementary matrix operations

There are three elementary matrix operations, each with its corresponding matrix form:

* Interchange two rows (or columns)
* Multiply each element in a row (or column) by a non-zero number
* Multiply a row (or column) by a non-zero number and add the result to another row (or column)

Row-wise (column-wise) elementary operations on a matrix are equivalent to pre-multiply (post-multiply) by the corresponding elementary matrices.

Elementary matrices are invertible. Through elementary matrix operations, any matrix can be converted to the standard form:

Here, and are product of a series of elementary matrices. It’s apparent that both and are invertible.

### Use the elementary matrix operations to computer the matrix’s inverse

A special case is when is a square matrix and is invertible, , then , which means ’s inverse is essentially the product of finite number of elementary matrices. This introduces a convenient approach to compute the inverse matrix.

If is invertible, then its inverse can be expressed by . Note if, then**.** This indicates the same chain of row-wise operations that converts to an identity matrix converts the identity matrix to its inverse !

See the following example:

Hence, the inverse of is:

## Rank of the matrix

Definition: for a given matrix , its maximum order (number of rows or columns) of sub-matrix with non-zero determinant is called ’s rank.

### Properties of matrix’s rank

* If is a square matrix, (full rank)
* Elementary matrix operations do not alter a matrix’s rank
* Column-wise and row-wise ranks are equal

### Use the elementary operations to determine a matrix’s rank

Given the fact that the elementary matrix operations do not alter a matrix’s rank, one can readily adopt this approach to compute a matrix’s rank. See the following example:

Hence.

# Vector space

## Maximal linearly independent subset

Given two vector sets , :

* If any in can be expressed as a linear combination of s in , and vice versa, then the two sets are equivalent
* If any in can be expressed as a linear combination of s in , and , then s in must be linearly dependent
* Assume . If i) all in are linearly independent and ii) any in can be written as a linear combination of , then is one of ’s maximal linearly independent subset

## Rank of the vector set

The number of vectors in the maximal linearly independent subset is defined as the vector set’s rank. If two vector sets are equivalent, they have the same rank.

### Relation to the rank of matrix

Think of the matrix as a set to row (column) vectors, the rank of the matrix equals to that of the row (column) vector set.

## Vector space and the base

The base of a vector space is essentially the maximal linearly independent subset of all vectors in the vector space. The number of the vectors in the base is called the vector space’s dimension.

The determinant of the matrix formed by the base vectors is not 0. This is because of the equality between matrix’s rank and vector set’s rank.

## Inner product

Let denote the inner product of two vectors and .

### Properties of inner product

* =
* Cauchy-Schwarz inequality:. This is because (projection of along is shorter than the magnitude of )

# System of linear equations

## Possible cases of solutions

For a system of linear equations with equations and variables: , there are three possible cases:

* If , this system is inconsistent and there are no solutions
* If , there is only one solution
* If , there are infinite solutions

Here is the augmented matrix of the coefficient matrix .

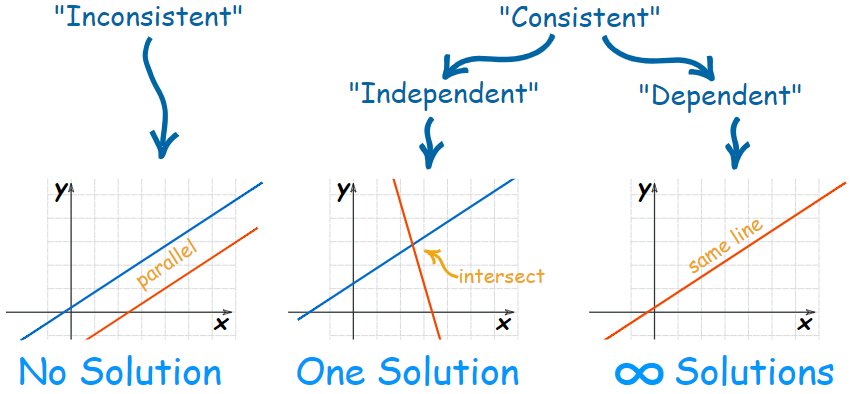


Figure 1. Three possible cases of solutions: i) no solution, ii) one solution, iii) infinity solutions

## Homogeneous systems

For a homogenous system with equations and variables: , the dimension of the solution set is . One special case is when , there is only one solution () and . In order to have non-zero solutions, .

### Examples

* Problem: solve the following system of linear equations:

Solution: the coefficient matrix . Hence, and the dimension of the solution set is . Let and be the free variables, the solutions can be expressed as:

## Non-homogeneous systems

Let be one special solution of the non-homogenous system with equations and variables: , and be the solution set of its homogenous counterpart , then the solution of is .

# Matrix similarity

## Eigenvalues and eigenvectors

Definition: let bea square matrix, if there exist a number and a non-zero vector that satisfies , then and are called ’s eigenvalues and eigenvectors, respectively.

Reshaping the equation leads to . In order to have non-zero solutions , one must have :

Since is an -order polynomial of , an -order squared matrix always has complex eigenvalues.

### Properties of eigenvalues

Assume the eigenvalues of an -order square matrix are: , , …, , then:

* + Proof: . When , . Hence .
* The eigenvalues of are , , …,
* The eigenvalues of are , , …,
* If is invertible, the eigenvalues of are , , …,

### Properties of eigenvectors

* Eigenvectors associated with different eigenvalues are linearly independent
* Let be an -order square matrix ’s -degenerate eigenvalue, the dimension of the vector space span by the eigenvectors associated with is . Hence, an -order square matrix has at most eigenvectors

### Condition number

Definition: the condition number of a matrix is the ratio of its maximal and minimal eigenvalues.

The eigenvalues can be thought of as the matrix’s ability to “magnify” the eigenvectors. Hence, the condition number reflects the matrix’s maximum and minimum magnifications.

The matrix is said to be ill-conditioned if the condition number is large. If the coefficient matrix of the system of linear equations is ill-conditioned, the solutions are instable.

In reality, when dealing with numeric matrices, the condition number is a better gauge to test the singularity of a matrix than its determinant. For instance, has a tiny determinant, but it’s actually well-conditioned with . Check this [link](https://scicomp.stackexchange.com/questions/1329/does-a-tiny-determinant-imply-ill-conditioning-of-a-matrix) for details.

### Examples

* Problem: find the eigenvalues and eigenvectors of the following matrix:

Solution: first of all let’s compute ’s eigenvalues:

.

When :

When :

* Problem: find the steady-state of the Markov chain transition matrix:

Solution: the steady-state of the Markov chain transition matrix satisfies :

Hence, is ’s eigenvalue. Using lets us find the associated eigenvector:

Note, one can solve the equation directly and obtain the same solution.

## Similar matrices

Definition: let and betwo square matrices, if there exists an invertible -order square matrix so that, then and are called similar matrices, denoted as .

### Properties of similar matrices

* and , then
* If , then i) , ii) , and iii) and have the same eigenvectors

## Matrix diagonalization

The major problem of similar matrices is to find an invertible square matrix , so that is a diagonal matrix .

An -order square matrix to be diagonalizable iff has linearly independent eigenvectors. Then is diagonalized by its eigenvalues and eigenvectors:

Here , , …, are eigenvalues, , , …, are the associated linearly independent eigenvectors.

Note, matrix inversion and matrix diagonalization are different concepts. They do not have any direct connections!

### Examples

* Problem: assume annually 10% of the city population moves to the country yard. On the other hand, 20% of the country population moves to the cities. Will all the population be concentrated in the cities many years later?

Solution: let the initial city and country population be and , respectively. The transition matrix is therefore:

Let , the goal is to figure out . To solve this, it’s more convenient to diagonalize first.

It’s not hard to find’s two eigenvalues: and . The associated eigenvectors are and . Hence

As , . Hence, , i.e., the ratio between the city and country population reaches a steady , which is fairly counter-intuitive.

* Problem: solve the following system of differential equations:

Solution: the given system of differential equations can be expressed in the matrix form:

Let , can expressed as:

Plug back to the original system of differential equations, yielding:

Let , then the original system is reduced:

The new system is much easier to solve!