Linear algebra notes

Justin

# Determinant

Unlike matrix which is a table of numbers, a determinant is just one number:

Note a determinant always has the same number of columns and rows, in other words, it is a square shape.

## Cofactor expansion

Let be the minor of the determinant after deleting the i-th row and j-th column:

The associated cofactor is then defined by . It is found that can be expanded according to a certain column or row. For instance, the expansion according to the 1st column is the following:

## Triangle determinant (determinant of a triangle matrix)

If is the determinant of a triangle matrix, then:

## Properties

* A determinant and its transpose have the same value:
* A determinant multiplied by a number is equivalent to the scenario in which the elements in a certain row or column are all multiplied by , i.e.,

Note this property is different from matrix.

* Interchanging two columns or rows changes the determinant’s sign
* If all elements in one row (column) are times that of the other, the determinant is 0
* If all elements in one row (column) can be broken to the sum to two other numbers, , then:
* Multiplying one row (column) by and adding it to another row (column) does not change the value of the determinant

## [Cramer’s rule](https://en.wikipedia.org/wiki/Cramer%27s_rule)

One example, let:

Then the solutions are:

Cramer’s rule is quite computationally expensive and hence is not adopted when solving the linear equations numerically.

## [Relation to the area of a parallelogram or the volume of a parallelepiped](https://mathinsight.org/relationship_determinants_area_volume)

### Relation to the area of a parallelogram

The area of a parallelogram spanned by the vectors and is the magnitude of :

Assume bothand lie in the same plane so that , hence:

Therefore, the area of the parallelogram is given:

### Relation to volume of a parallelepiped

The volume of a parallelepiped spanned by the vectors , , and is the magnitude of :

Hence, the volume of a parallelepiped can be expressed as:

# Matrix

Unlike the determinant which is essentially a number, the matrix is a table of numbers. The identity matrix is equivalent to 1 in numbers.

## Addition and multiplication by numbers

Matrix addition and number multiplication have the following properties:

Note multiplying a matrix by a number is equivalent to multiplying each element of this matrix by this number. This is different from multiplying a determinant by a number.

## Matrix multiplication

Let , then each element in is the inner product of the row vectors in and column vectors in :

Here, and are the row and column vectors of and , respectively. Matrix multiplication has the following properties:

In addition, it’s possible that , even though and .

## Matrix transpose

Matrix transpose has the following properties:

Given that , the last property is exceptionally interesting. It can be generalized to finite number of matrices:

Both and are symmetric matrices.

Any square matrix can be decomposed to the sum of a symmetric and an anti-symmetric matrix:

## Matrix inverse

Definition: if , then is invertible and the inverse of is.

For to be invertible, the sufficient and necessary condition is . Under such a circumstance:

Here is’sadjugate matrix. If , then has the following expression:

Note is the transpose of the matrix formed by the cofactors of each element in .

Matrix inverse has the following properties:

Note the last one is very similar to matrix transpose. It can be quickly proved in the following:

Similar to matrix transpose, this property can be generalized to finite number of matrices:

## Block matrix

### Block matrix inverse

let , then .

### Block matrix multiplication

In addition, let , then . Here . In other words, the rule is the same as ordinary matrix multiplication.

### Some special cases

If , , then the following rules hold:

It’s not hard to see that these rules are the same as ordinary matrix opertions.

## Elementary matrix operations

There are three elementary matrix operations, corresponding to three elementary matrices:

* Interchange two rows (or columns)
* Multiply each element in a row (or column) by a non-zero number
* Multiply a row (or column) by a non-zero number and add the result to another row (or column)

Row-wise (column-wise) elementary operations on a matrix are equivalent to pre-multiply (post-multiply) by the corresponding elementary matrices.

Through elementary matrix operations, any matrix can be converted to the standard form:

Here, and are multiplication of a series of elementary matrices that correspond to row-wise and column-wise elementary operations, respectively. It’s not hard to see that both and are invertible.

### Use the elementary operations to determine a matrix’s inverse

A special case is when is a square matrix and is invertible, , then , which means the invertible matrix is essentially the multiplication of finite number of elementary matrices. This introduces a convenient approach to compute the inverse matrix. If is invertible, then its inverse can be expressed by . Note:

This indicates the same chain of row-wise operations that converts to an identity matrix converts the identity matrix to its inverse !

See the following example:

Hence, the inverse of is:

## Rank of the matrix

Definition of the matrix’s rank: for a given matrix , its maximum order (number of rows or columns) of sub-matrix with non-zero determinant is called ’s rank.

Properties of matrix’s rank:

* If is a square matrix, (full rank)
* Elementary matrix operations do not change a matrix’s rank

Note for any matrix, its column-wise rank equals to its row-wise rank.

### Use the elementary operations to determine a matrix’s rank

Given the fact that the elementary matrix operations do not change a matrix’s rank, one can adopt this approach to determine a matrix’s rank. See the following example:

Hence.

# Vector space

## Maximal linearly independent subset

Given two vector sets , :

* If any can be expressed as a linear combination of , and vice versa, then the two sets are equivalent
* If any can be expressed as a linear combination of , and , then must be linearly dependent
* Assume . If i) is linearly independent and ii) can be expressed as a linear combination of , then is one of ’s maximal linearly independent subset

## Rank of the vector set

The number of vectors in the maximal linearly independent subset is defined as the vector set’s rank. If two vector sets are equivalent, they share the same rank.

### Relation to the rank of matrix

Think of the matrix as a set to row (column) vectors, the rank of the matrix equals to that of the row (column) vector set.

## Vector space and the base

The base of a vector space is essentially the maximal linearly independent subset of all vectors in the vector space. The number of the vectors in the base is called the vector space’s dimension.

The determinant of the matrix formed by the base vectors is not 0. This is because of the equality between matrix’s rank and vector set’s rank.

## Inner product

Define the inner product of two vectors and as , it has the following properties:

* =
* Cauchy-Schwarz inequality:. This is because (projection of along is shorter than the magnitude of )

# System of linear equations

## Possible cases of solutions

For a system of linear equations with equations and variables: , there are three possible cases:

* If , this system is inconsistent and there are no solutions
* If , there is only one solution
* If , there are infinite solutions

Here is the augmented matrix of the coefficient matrix .

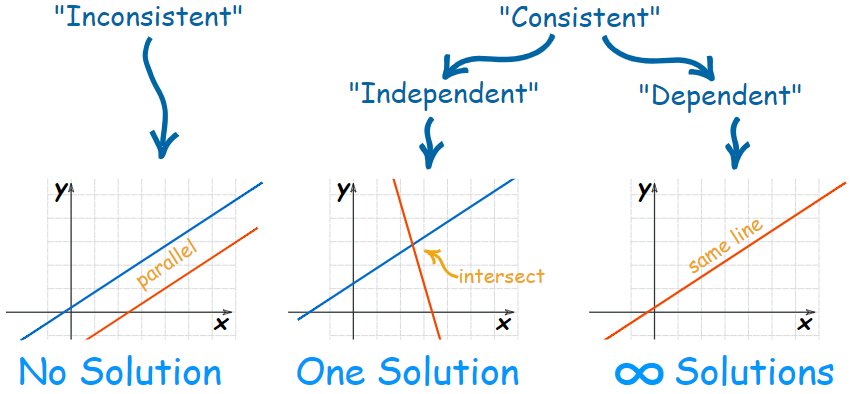


Figure . Three possible cases of solutions: i) no solution, ii) one solution, iii) infinity solutions

## Homogeneous systems

For a homogenous system with equations and variables: , the dimension of the solution set is . One special case is when , there is only one solution () and . In order to have non-zero solutions, .

### Example

Solve the following system of linear equations:

Solution: the coefficient matrix . Hence, and the dimension of the solution set is . Let and be the free variables, the solutions can be expressed as:

## Non-homogeneous systems

Let be one special solution of the non-homogenous system with equations and variables: , and be the solution set of its homogenous counterpart , then the solution of is .

# Matrix similarity

## Eigenvalues and eigenvectors

Let bea square matrix, if there exist a number and a non-zero vector that satisfies , then and are called ’s eigenvalues and eigenvectors.

Reshaping the equation leads to . In order to non-zero solutions, :

Since is an -order equation for , an -order squared matrix has eigenvalues.

### Properties of eigenvalues

Assume the eigenvalues of an -order square matrix are: , , …, , then:

* + Proof: . When , . Hence .
* The eigenvalues of are , , …,
* The eigenvalues of are , , …,
* If is invertible, the eigenvalues of are , , …,

### Properties of eigenvectors

* Eigenvectors associated with different eigenvalues are linearly independent
* Let be an -order square matrix ’s -degenerate eigenvalue, the dimension of the vector space formed by the eigenvectors associated with is less than . Hence, an -order square matrix has at most eigenvectors

### Example 1

Find the eigenvalues and eigenvectors of the following matrix:

Solution: .

When :

When :

### Example 2

Find the steady-state of the Markov chain transition matrix:

Solution: the steady-state of the Markov chain transition matrix satisfy :

Hence, is ’s eigenvalue. Using let us find the associated eigenvector:

Note, one can solve the equation directly and achieve the same solution.